Advanced Topics in Combinatorics **Summer 2018**

Lecture 1: Generating Functions

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1.1 Introduction

The *generating function* is a very powerful tool in combinatorics, and is the first one we will be introducing in this class, mostly because it has applications throughout combinatorics and will be very useful in proving results in Ramsey theory, partition theory, code theory, and more. It is also worth getting to know early, because at first it may be difficult to gain intuition as to why we use them, and just how powerful a technique they are.

Definition 1.1. An **(ordinary) generating function** for a series $a_0, a_1, a_2, \dots$ is the power series

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

As an example, the generating function for the sequence $1, 2, 1, 0, 0, 0, 0, \dots$ is $1 + 2x + x^2 = (1+x)^2$, and that for the sequence $1, 1, 1, 1, 1, 1, \dots$ is $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$. Why is such a concept useful? In many cases, even if there are infinitely many nonzero entries to the series, we can reduce this infinite power series to a closed, finite form; and, as it turns out, we can **add** and **multiply** generating functions for different power series in a meaningful, natural way (as well as divide, scale, differentiate, and any other meaningful operations on polynomials). This becomes extremely useful, as we will show with examples later in the lecture. A particular simple example of where you can see generating functions pop up is in the binomial theorem, which I'm sure most of you know.

Theorem 1.2 (Binomial Theorem)**.** *For an integer n and variables x, y, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$*

Now, if y is a variable, note that this actually gives a generating function for some series; it is indeed finite, but a generating function nonetheless. Now, at this point, it may seem that the connection to generating functions is rather fuzzy and not useful, but it becomes more clear when you consider a generalized version of the binomial theorem, in which the exponent is replaced by any real number. The details of why it is defined this way are not worth getting into, but either way, the series becomes infinite and you can see why generating functions are a natural way to express such a sum, and give a reason to examine the underlying sequence. Note that a binomial $\binom{j}{k}$ for a noninteger j and integer k is defined as $\frac{j(j-1)(j-2)\cdots(j-k+1)}{k!}$.

Theorem 1.3 (Newton's generalized binomial theorem)**.** *For any real number j and variables x, y, we have that $(x+y)^j = \sum_{k=0}^{\infty} \binom{j}{k} x^{j-k} y^k$.*

These are examples where you can see generating functions is everyday theorems, but why are generating functions themselves a powerful tool in solving problems? To give some intuition behind this, I will give a rather elongated problem in which generating functions turn out to be extremely useful: deriving a explicit form for the Catalan numbers. This technique will prove invaluable in general for solving recurrences.

1.2 Catalan numbers: back to the quadratic formula?

Formally, the Catalan numbers are a sequence $C_0, C_1, C_2, \ldots, C_n$ which can be described by the following recurrence $C_0 = 1, C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}$.

Now, this is a rather dry formalism and may seem like a contrived recurrence that comes out of nowhere, but the Catalan numbers turn out to have a number of very interesting combinatorial applications. For example, the Catalan numbers are exactly the number of ways to travel from $(0,0)$ to (n,n), only moving unit distances up or to the right, and staying below the line $y = x$. Alternatively, the Catalan numbers are exactly the number of ways to write a series of n open and close parentheses that make sense; i.e., there can never be more close parantheses than open parentheses that have appeared before them (Why is this the same as the previous definition?). The Catalan numbers turn out to appear in a lot of places[1]. As one might imagine, because of the ubiquity of the Catalan numbers, it would be useful to derive an explicit formula for them, rather than the rather cumbersome recurrence above. How are we going to do this? *Generating functions*, of course!

What is this generating function? We know that in general, from our definition, the generating function for the Catalan numbers is of the form $\sum_{i=0}^{\infty} C_n x^n$. However, the utility of generating functions comes from writing this is a more compressed form, and to this end we will use the recurrence relation of the Catalan numbers.

Claim 1.4. *The generating function $\mathcal{C}(x)$ for the Catalan numbers satisfies $\frac{1}{x}\mathcal{C}(x) - 1 = \mathcal{C}^2(x)$.*

Proof: We're going to use our recurrence relation to rewrite the generating function. To this end, consider a slight alteration to our traditional generating function, the sum

$$\sum_{n=0}^{\infty} C_{n+1} x^n \tag{1.1}$$

Using our recurrence, (1.1) is equivalent to $\sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} C_i C_{n-i} \right) x^n$. But note that this is actually just another way of writing $\mathcal{C}(x)^2$, since each term of this sequence is exactly equivalent to those you would get if you squared our original generating function. However, we also know that (1.1), if we multiply by x, is just our original generating function, except without the $C_0 x^0$ term: in other words, it is $\frac{1}{x}(\mathcal{C}(x) - C_0) = \frac{1}{x}(\mathcal{C}(x) - 1)$. Thus, since both of our derivations must be equal since they are from the same original equation, we have that $\frac{1}{x}(\mathcal{C}(x) - 1) = \mathcal{C}(x)^2$, as desired. ■

Now comes the weird part. We *know* how to solve this: use the quadratic formula! In particular, multiplying by x, we have that $x\mathcal{C}(x)^2 - \mathcal{C}(x) + 1 = 0$, so, using our favorite formula, $\mathcal{C}(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$. Now, which solution is it? To figure this out, as with many recurrences, we use the base case: $C_0 = 0$. Now, note that if we plug in $x = 0$ into our generating function $\sum_{n=0}^{\infty} C_n x^n$, we should just get the C_0 term, since all others go to 0. However, if we plug in 0 into the positive square root, we get $\frac{2}{0} = \infty$, which is clearly impossible (with the negative square root, we get $\frac{0}{0}$, which is indeterminate, and still could be the correct solution). Thus, the solution we are looking for is the negative square root: our generating function $\mathcal{C}(x) = \frac{1-\sqrt{1-4x}}{2x}$.

[1] For those interested, Richard Stanley, a famous combinatorist at MIT, wrote up a compendium of hundreds of interpretations/ways to derive the Catalan numbers which can be found here: `http://www-math.mit.edu/~rstan/ec/catadd.pdf`

From here, we're going to use a theorem I mentioned earlier, Newton's generalized binomial theorem (it wasn't just a digression!), to derive an explicit form generating function for the Catalan numbers.

Proposition 1.5 (Explicit form of Catalan Numbers [2])**.** *For all* $n \geq 0$, $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Proof: Using the generalized binomial theorem, we can write $1-(1-4x)^{1/2}$ as $1-\sum_{n=0}^{\infty}\binom{0.5}{n}(-4)^n x^n$. The first step to simplifying this is expanding $\binom{0.5}{n}$ as $\frac{0.5(0.5-1)\ldots(0.5-n+1)}{n!}$. Multiplying this by $(-4)^n$, we get

$$\frac{2(2-4)\ldots(2-4n+4)}{n!}(-1)^n = -\frac{2(2)(6)\ldots(4n-6)}{n!} = -\frac{2^n(1)(3)\cdots(2n-3)}{n!}$$
$$= -\frac{2^n(n!)(1)(3)\cdots(2n-3)}{n!\cdot n!} = -\frac{(2n)!}{n!\cdot n!\cdot(2n-1)} = -\frac{1}{2n-1}\binom{2n}{n}.$$

Thus, $1-(1-4x)^{1/2} = 1-\sum_{n=0}^{\infty}-\frac{1}{2n-1}\binom{2n}{n}x^n = \sum_{n=1}^{\infty}\frac{1}{2n-1}\binom{2n}{n}x^n$. This gives us that $\mathcal{C}(x) = \frac{1}{2x}\sum_{n=1}^{\infty}\frac{1}{2n-1}\binom{2n}{n}x^n = \sum_{n=0}^{\infty}\frac{1}{4n+2}\binom{2n+2}{n+1}x^n$. At this point, we are done, since $C_n = \frac{1}{4n+2}\binom{2n+2}{n+1}$, although we can, as it turns out, further simplify this (in a non-interesting way) to $\frac{1}{n+1}\binom{2n}{n}$, the more common explicit formula. Then, we are done! ■

1.3 Fibonacci: where did the $\sqrt{5}$ come from?

Now, the methods that we used to find an explicit form for the Catalan numbers are not unique! They can generally be applied to solve many recurrences; multiply your given generating function by x, plug in the recurrence, and derive an equation for the generating function in terms of it and x. Using the quadratic equation or similar techniques, solve for the generating function as a fraction in terms of x, and use series expansions (the binomial theorem, Taylor expansions that you'll use in calculus, or others), to derive an explicit form for the generating function and thus the sequence as a whole.

An example of this can be seen with the Fibonacci sequence, which satisfies $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1}+F_{n-2}$, for which we'll write the generating function as $\mathcal{F}(x) = \sum_{n=0}^{\infty}F_n x^n$. Then, using our recurrence, $\frac{1}{x}\mathcal{F}(x) = \sum_{n=1}^{\infty}F_n x^{n-1} = \sum_{n=1}^{\infty}F_{n-1}x^{n-1} + \sum_{n=1}^{\infty}F_{n-2}x^{n-1} = \mathcal{F}(x)+x\mathcal{F}(x)+F_{-1}$. Thus, we have that $\frac{1}{x}\mathcal{F}(x) = \mathcal{F}(x)+x\mathcal{F}(x)+1$. Multiplying by x and solving for $\mathcal{F}(x)$, we get that $\boxed{\mathcal{F}(x) = \frac{x}{1-x-x^2}}$. We won't do the series expansion explicitly here, but it turns out that this series expansion gives that $F_n = \frac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^n\sqrt{5}}$!! This is quite extraordinary [3], and may beg the question: where does such a result come from, intuitively? The main explanation I can give is a shaky one; the roots of the polynomial in the denominator of the generating function, $1-x-x^2$ are exactly $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$, and it turns out that we do a partial fraction decomposition based on these roots to derive the explicit form generating function.

[2] This proof is adapted from that of Mike Spivey

[3] At first it might not be entirely obvious that this gives a result that is an integer, until you realize that only the $\sqrt{5}$ terms remain in the expansion of the numerator, and dividing by the $\sqrt{5}$ in the denominator gives at least a rational answer; showing that this is an integer is another story.

Either way, from these two results, you can see that using generating functions to solve recurrences is an extremely powerful tool, and coupled with characteristic polynomials, which we may talk about in a further lecture, can solve most recurrences that you'll encounter.

1.4 Generating Functions meet Cookies

A common problem involving generating functions involves the fact that their product is meaningful in the context of a counting problem. In particular, let's say that $p_0, p_1, p_2, p_3, \ldots$ are the number of ways of choosing n elements from some set P, and $q_0, q_1, q_2, q_3, \ldots$ are the number of ways of choosing n elements from some set Q. Clearly, we can write these sequences as $\mathcal{P}(x) = \sum_{n=0}^{\infty} p_n x^n$ and $\mathcal{Q}(x) = \sum_{n=0}^{\infty} b_n x^n$. Now, let r_n be the number of ways of choosing n elements from either P or Q; clearly r_n is just $\sum_{i=0}^{n} p_i q_{n-i}$, which gives that $\mathcal{R}(x) = \mathcal{Q}(x)\mathcal{P}(x)$. This identity shows that, in counting problems, we can multiply generating functions associated with choosing objects from particular sets to get the combined solution.

This fact turns out to be extremely powerful. For example, consider the following problem involving different types of cookies: oreos, milanos, chips ahoy, and thin mints (my favorite!).

Problem 1.4.1. *Let $p(n)$ be the number of ways to choose n oreos, milanos, chips ahoy, and thin mints, subject to the following constraints: the number of oreos we take must be a multiple of 7, we can only take* 6 *milanos, we can take only* 2 *chips ahoy, and the number of thin mints must be odd. What is $p(n)$ for all n?*

To solve this problem, we will find a generating function for all of the different quantities we are choosing, and then multiply them. Amazingly, this will give us a simple answer to what seems to be a pretty ugly counting problem – generating functions eliminates all of the tediousness.

Solution: First, let the number of the oreos, milanos, chips ahoy, and thin mints, respectively, be a, b, c, d. Then, our constraints are that $a \equiv 0 \bmod 7$, $b \le 6$, $c \le 2$, $d \equiv 1 \bmod 2$. We want to write generating functions for all of these. Consider the oreos first; since $a \equiv 0 \bmod 7$, the associated "possibility" sequence is $0, 7, 14, 21, 28$. If we write our generating function as taking a 1 if a quantity is possible and 0 otherwise, this gives us a generating function of $\mathcal{A}(x) = \sum_{n=0}^{\infty} x^{7n} = \frac{1}{1-x^7}$. Similarly, for milanos, the possibilities are $0, 1, 2, 3, 4, 5, 6$, giving a generating function of $\mathcal{B}(x) = \sum_{n=0}^{6} x^n = \frac{1-x^7}{1-x}$. Deriving a similar generating function for chips ahoy and thin mints gives $\mathcal{C}(x) = 1 + x$ and $\mathcal{D}(x) = \sum_{n=0}^{\infty} x^{2n+1} = \frac{x}{1-x^2}$. Now, we use the fact that we can multiply; the total possibilities generating function $\mathcal{T}(x)$ is just $\mathcal{A}(x)\mathcal{B}(x)\mathcal{C}(x)\mathcal{D}(x) = \frac{1}{1-x^7} \cdot \frac{1-x^7}{1-x} 1 + x \cdot \frac{x}{1-x^2} \Rightarrow \mathcal{T}(x) = \frac{x}{(1-x)^2}$. To express this in typical form for a generating function, I will take a small digression into calculus, using the fact that $\frac{1}{1-x}$ has expansion $1 + x + x^2 + \cdots$ and that the derivative of $\frac{1}{1-x}$ is $\frac{1}{(1-x)^2}$. Using the fact that derivatives apply equal well to power series (a fact that I have not proven), this gives that $\frac{1}{(1-x)^2}$ can be written as $\sum_{n=0}^{\infty} (n+1)x^n$, and thus our desired generating function is $\frac{x}{(1-x)^2} = \boxed{\sum_{n=1}^{\infty} nx^n}$, which gives us that the number of ways to choose n cookies among the four types is exactly n! Surprising, huh? ■

In general, this kind of method is very powerful when we are asked to choose items with respect to some requirement, and can make our lives a whole lot easier when doing many types of combinatorics problems!

1.5 Exponential Generating Functions and Bell numbers

This section will introduce a new concept, called the exponential generating function. The main purpose of this sort of function is to solve a counting problem in which our goal is to count the number of ways the n-element set can take some structure; for ordinary generating functions, our objects were indistinguishable, in some sense (although multiplication sometimes allowed us to change this), while with exponential generating functions our objects are necessarily ordered. This is a rather vague prerogative, but it will become more clear as we work through examples. Formally, an exponential generating function is defined as follows:

Definition 1.6. An **exponential generating function** for a series $a_0, a_1, a_2, \ldots$ is the power series

$$a_0 + a_1 x + \frac{a_2 x^2}{2} + \frac{a_3 x^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$$

.

It helps to work through some trivial examples; if the series we are considering is $1, 1, 1, 1, \ldots$, then the exponential generating function is just $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$, by definition. In many cases, however, it helps to think of our series rather as a set; the first example described the general set, the second the trivial 0-element set. If we want an even-sized set, or a series $1, 0, 1, 0, 1, \ldots$, then the exponential generating function is $\frac{e^x + e^{-x}}{2}$, which is the definition of hyperbolic cosine. You can see that we're getting some pretty weird functions, and this helps in many cases in which recurrences are strange or the sequences/structures we are studying require us to impose structure on a set.

The additive rule we used frequently for ordinary generating functions still applies to exponential generating functions, but this is just about the only rule that still applies. Multiplication is now different; with ordinary generating functions, we used that $r_n = \displaystyle\sum_{i=0}^{n} p_i q_{n-i}$. However, if we multiply two exponential generating functions, we instead get that $r_n = \displaystyle\sum_{i=0}^{n} \binom{n}{i} p_i q_{n-i}$; so, for future reference, when we say that $\mathcal{R}(x) = \mathcal{P}(x)\mathcal{Q}(x)$ for exponential generating functions, this is the implication about the underlying sequences.[4]

Using these definitions, I want to prove a famous formula for the Bell numbers B_n, which represent the number of partitions of a set of size n. For example, for the three element set $\{x, y, z\}$, the possible partitions are $\{\{x\}, \{y\}, \{z\}\}, \{\{x, y\}, \{z\}\}, \{\{x\}, \{y, z\}\}, \{\{x, z\}, \{y\}\}, \{\{x, y, z\}\}$, so $B_3 = 5$. It turns out that the Bell numbers satisfy the following explicit formula:

Theorem 1.7 (Dobinski's Formula)**.** *The nth Bell number satisfies* $B_n = \frac{1}{e} \displaystyle\sum_{k=0}^{\infty} \frac{k^n}{k!}$.

The fact that we have a infinite sum and a $\frac{1}{e}$ is a pretty good indicator in general that we'd be using exponential generating functions, but we'll start from scratch.

[4]For further reference, shifting to the left or right in the underlying sequence can no longer be achieved by substituting x^k; instead differentiating and integrating are the corresponding operations. We'll try to avoid calculus as much as possible, however.

Proof: First, let's think about finding a recurrence relation for B_{n+1}. To do this, think about what happens once we fix the subset that the first element of our set is in; it can be of any size $n+1 \geq j \geq 1$, for which there are $\binom{n}{j-1}$ ways of choosing the remaining $j-1$ elements. Summing over all such j, we get the recurrence $B_{n+1} = \sum_{j=1}^{n+1} \binom{n}{j-1} B_{n+1-j}$, or as it is more typically written $\sum_{k=0}^{n} \binom{n}{k} B_k$, if we switch around some indices.

From here, we are going to prove the following lemma, which will help us derive the desired result.

Lemma 1.8. *The exponential generating function $\mathcal{B}(x)$ for the Bell numbers is e^{e^x-1}.*

Proof: The first step is a bit tricky, and involves the recurrence $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$ that we just derived. Remember that we can actually write the multiplication of two exponential generating functions p and q as defining a new sequence such that $r_n = \sum_{k=0}^{n} \binom{n}{k} p_k q_{n-k}$. In the above, if we let $p_k = B_k$ and $q_k = 1 \forall k$, and the exponential generating function for the Bell numbers be $\mathcal{B}(x)$, we get that $e^x B(x) = \frac{dB}{dx}$. The e^x comes from the fact that this is the generating function for the sequence of all 1's, which is the q_k in this case, and the $B'(x)$ comes from the fact that our multiplication shifts the values in the series one over, which I briefly mentioned was equivalent to differentiating (this is not so difficult to show, but we will not get into it right now). Either way, from here, moving terms, we get that $\frac{dB}{B} = e^x dx$, which, integrating, gives that $\ln(B) = e^x + C \Rightarrow B(x) = e^{e^x + C}$. Using the base case that $B(0) = B_0 = 1$, since this gives us the first term of the series, we have that $C = -1$, so we get that the exponential generating function for the Bell numbers is e^{e^x-1}, as desired. ■

With this in hand, we now need to just expand this exponential generating function to get our desired explicit formulation. In particular, $\mathcal{B}(x) = \frac{1}{e} e^{e^x}$; let's first expand e^{e^x}. We'll have to use the so-called Maclaurin series (which you'll learn in calculus) for $e^x = \sum_{k=0} \frac{x^k}{k!}$. Just plugging this in, this reduces our generating function (disregarding the factor of $\frac{1}{e}$ for now) to $\sum_{k=0}^{\infty} \frac{(e^x)^k}{k!}$. From here, plugging in the Maclaurin expansion for each individual term in the sum, we get that $e^{kx} = \sum_{n=0}^{\infty} \frac{k^n x^n}{n!}$, so the generating function can be expanded out as

$$\frac{1}{e} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{k^n x^n}{n! \cdot k!} = \frac{1}{e} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{k^n}{k!} \right) \frac{x^n}{n!}.$$

Since we of course also know that $\mathcal{B}(x) = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}$, this gives that $B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}$, as desired.

■

1.6 Let's solve some HMMT Problems!

I thought we'd finish with some HMMT problems that are easily solvable using techniques we've addressed up to this point. We've only begun to scratch the surface of generating functions and their utility, however, and I'd recommend that, if interested, you delve more deeply into the additive, multiplicative, and differentiable properties I briefly mentioned in this lecture, and read some resources that will help you develop an even more rigorous theory of generating functions. Regardless, here goes!

Problem 1.6.1 (HMMT 2007, Problem 9, Combinatorics)**.** *Let S denote the set of all triples (i,j,k) of positive integers where $i+j+k=17$. Compute* $\sum_{(i,j,k)\in S} ijk$.

Solution: This problem may look intimidating, but we're going to solve for the general case, not just $i+j+k=17!$. Let $s_n = \sum_{i+j+k=n} ijk$. We want to find the generating function $\sum_{n\geq 0} s_n x^n$. But note that this generating function simply corresponds to cubing the generating function $\sum_{n\geq 0} nx^n$, because cubing this function causes the coefficient of x^n to be exactly the sum of the products of numbers that add up to n. Now, let $S = \sum_{n\geq 0} nx^n$. Multiplying by x and subtracting this from S, we get that $S - xS = \sum_{n\geq 1} x^n = \frac{x}{1-x}$, by the sum of a geometric series. This gives that $S = \frac{x}{(1-x)^2}$, and thus our generating function is $\left(\frac{x}{(1-x)^2}\right)^3 = \frac{x^3}{(1-x)^6}$. From here, using the generalized binomial theorem, we have that $(1-x)^{-6} = \sum_{k=0}^{\infty}\binom{-6}{k}x^k(-1)^k$. Then, we expand $\binom{-6}{k} = \frac{-6(-7)\cdots(-6-k+1)}{k!}$; multiplying $(-1)^k$ gives this to be $\frac{(k+5)\cdots(6)}{k!} = \binom{k+5}{5}$. Thus, our generating function is $x^3\sum_{k=0}^{\infty}\binom{k+5}{5}x^k = \sum_{k=3}^{\infty}\binom{k+2}{5}x^k$. Thus, we have that $s_k = \binom{k+2}{5}$, so our desired answer is $\boxed{\binom{19}{5} = 11628}$, which is correct! ■

A problem that also falls easily in a similar way to generating functions arguments (quite similar to problems that we've discussed), also from the HMMT, is the following. See if you you can solve it!

Problem 1.6.2 (HMMT 2008, Combinatorics, Problem 10)**.** *Determine the number of 8-tuples of nonnegative integers $(a_1,a_2,a_3,a_4,b_1,b_2,b_3,b_4)$ satisfying $0\leq a_k\leq k$, for each $k=1,2,3,4$, and $a_1+a_2+a_3+a_4+2b_1+3b_2+4b_3+5b_4=19$.*

Now, finally, we end with a problem that doesn't immediately look like it can be solved with generating functions, but turns out to fall immediately to a generating function that we derived earlier in this lecture!

Problem 1.6.3 (HMMT 2008, Algebra, Problem 10)**.** *Evaluate the infinite sum*

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{5^n}.$$

Solution: Recall that we know the generating function for the Catalan numbers, which has the form $\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n$ and $\frac{1-\sqrt{1-4x}}{2x}$. If we multiply the Catalan number's equation by x and take a derivative, we get the generating function $G(x) = \sum_{n=0}^{\infty}\binom{2n}{n}x^n$, which is what we are looking for (we simply want to plug $\frac{1}{5}$ into this generating function). Doing the same to our other known explicit form for the Catalan generating function, we differentiate $x\left(\frac{1-\sqrt{1-4x}}{2x}\right) = \frac{1}{2}(1-\sqrt{1-4x})$, which gives $G(x) = \frac{1}{\sqrt{1-4x}}$. Plugging in $\frac{1}{5}$, this gives an answer to the original problem of $\frac{1}{\sqrt{1-\frac{4}{5}}} = \boxed{\sqrt{5}}$, which is indeed correct! ■

1.7 Practice and Challenge Problems

1.7.1 Basic Problems

Problem 1.7.1. *Find the generating function for the sequences* $1, 1, 1, 1, 0, 0, 0, 0, \ldots$ *and* $1, 7, 21, 35, 35, 21, 7, 1, 0, 0, \ldots$. *Simplify your answers as much as possible.*

Problem 1.7.2. *In how many ways can we make \$1 from pennies, nickels, dimes, and quarters?*

Problem 1.7.3. *Prove that the generating function* $\prod_{n\geq 1}(1-x^n)^{-\mu(n)/n} = e^x$, *for* $\mu(n)$ *the Mobius function defined as* $\mu(1) = 1$, $\mu(n) = 0$ *if* n *is divisible by the square of an integer greater than one, and* $\mu(n) = (-1)^r$ *if* n *is the product of* r *distinct primes. Hint: take logs! This is a beautiful, surprising result, which does require some calculus.*

Problem 1.7.4. *Solve the linear recurrence* $a_n - 4a_{n-1} + 4a_{n-2} = 0, n \geq 2$, $a_0 = 1, a_1 = 4$, *using generating functions.*

Problem 1.7.5. *Using* 1×2 *dominoes that we can place horizontally or vertically, in how many ways can we tile a* $2 \times n$ *strip? These numbers may look familiar!*

As a note, the following theorem is true in the general case:

Theorem 1.9 (Kasteleyn, 1961)**.** *Let* $Q_{m,n}$ *be a rectangular grid with* m *even. Then the number of* 1×2 *domino tilings of* $Q_{m,n}$ *is* $\displaystyle\prod_{k=1}^{0.5m}\prod_{j=1}^{n} 2\sqrt{\cos^2\left(\frac{k\pi}{m+1}\right) + \cos^2\left(\frac{j\pi}{n+1}\right)}$.

Isn't that an astonishing result! We only ask you to prove the $2 \times n$ case, but it's still a fascinating problem and one very approachable using generating functions (in fact, the proof of the above relies on constructing such a generating function).

1.7.2 Competition Problems

Problem 1.7.6. (Romania 2003) *How many* n*-digit numbers, whose digits are in the set* $\{2, 3, 7, 9\}$, *are divisible by* 3*?*

Problem 1.7.7. (IMO 1995) *Let* p *be an odd prime. How many* p*-element subsets* A *of* $\{1, 2, \ldots, 2p\}$ *are there, the sum of whose elements is divisible by* p*?*

Problem 1.7.8. (IMO Shortlist 1998) *The sequence* $0 \leq a_0 < a_1 < a_2 < \cdots$ *is such that every nonnegative integer can be uniquely expressed as* $a_i + 2a_j + 4a_k$, *where* i, j, k *are not necessarily distinct. Find* a_{1998}.

1.7.3 Research Problems

Generating functions are ubiquitious in research, especially in enumerative combinatorics. I give an example below of a problem that is actually related to a famous NP-complete problem that baffled researchers for decades, for which the smaller cases can be solved using techniques of generating functions. See if you can figure out how![5]

Problem 1.7.9 (Subset sum problem)**.** *Given a list of integers, is there a nonempty subset of these integers whose sum is a particular integer* k*? If so, find it.*

[5] Note that this isn't supposed to be especially elegant, but is a way that you could do this problem for small cases. Pretty cool that something you can learn in high school is still used in modern algorithms problems!